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Representations of the quantum algebra $U_q(su_{1,1})$

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Abstract. All irreducible representations of the quantum group $U_q(su_{1,1})$ are given. They are determined by two complex numbers. Infinitesimally unitary representations are seperated from the set of irreducible representations. It is shown that the symmetric operators of irreducible infinitesimally unitary representations corresponding to generators of the Lie algebra $su_{1,1}$ admit self-adjoint extensions.

1. Introduction

Quantum groups and algebras appeared in the quantum method of the inverse scattering problem. They are of great importance for applications in quantum integrable systems, in quantum field theory, and in statistical physics. To apply them it is neccessary to have a well developed theory of their representations. Representations of the simplest quantum algebras are of great significance for applications. There is, more or less, a clarity about finite-dimensional representations of quantum groups and algebras: inequivalent irreducible representations are classified, uniqueness of highest weights has been proven, the relation to irreducible representations of Lie groups and algebras has been shown, and so on (Rosso 1988). It is not so clear with infinitesimally unitary representations of 'non-compact' quantum algebras and with unitary representations of non-compact quantum groups.

In this paper we deal with infinite-dimensional representations of the quantum algebra $U_q(su_{1,1})$. Such representations were considered by Klimyk and Groza (1989), Masuda *et al* (1990) and Vaksman and Korogodsky (1990). A review of these results has been given by Klimyk *et al* (1990). The representations considered in these papers are given by a complex number λ and by a number $\epsilon \in \{0, 1/2\}$. They are a q-analogue of the corresponding representations of the classical Lie group SU(1,1). These representations do not exhaust all irreducible (and unitary) representations of $U_q(su_{1,1})$.

There are representations of the Lie algebra su(1,1) which lead to representations of the universal covering group $\widetilde{SU}(1,1)$ for the Lie group SU(1,1). Such representations are given by two complex numbers. In this paper we consider a q-analogue of these representations for $U_q(su_{1,1})$. Here the strange series of infinitesimally unitary representations of $U_q(su_{1,1})$ appears. They are given by two continuous parameters. These representations disappear for the Lie algebra su(1,1) (when q=1). Clearly, all representations in the papers mentioned above are a part of the set of our representations. Let us emphasize that representations of $U_q(su_{1,1})$ are related to the q-oscillators (Kulish and Damaskinsky 1990).

We also consider the problem of self-adjointness of representation operators. This problem is solved for the Lie group SU(1,1) and for other semi-simple Lie groups. There is the well developed theory of self-adjointness of representation operators for Lie algebras (Barut and Raczka 1977). For infinite-dimensional representations of quantum algebras no results in this area existed. As we shall see, the situation for the quantum case is different from the case of Lie groups. For example, eigenvectors of the representation operator $T(H)$ (see section 2) are not analytical vectors of an irreducible infinite-dimensional representation of the quantum algebra $U_q(su_{1,1})$. Nevertheless, the operators $T(E_+ + E_-)$ and $T(iE_+ - iE_-)$, as in the classical case, admit self-adjoint extensions.

In the classical case, the discrete series representations are realized in the Bargmann space. The discrete series representations of $U_q(su_{1,1})$ can be realized in the q-analogue of the Bargmann space (Bracken *et al* 1991). This realization may be used to study self-adjointness of operators $T(E_+ + E_-)$ and $T(iE_+ - iE_-)$. However, we use the method of Jacobi matrices because it can be applied to other series of representations of $U_q(su_{1,1})$.

In section 2 we define the quantum algebra $U_q(su_{1,1})$. In sections 3 and 4 the standard representations $T_{a\epsilon}$ of $U_q(su_{1,1})$ are constructed. The classification of irreducible representations of $U_q(su_{1,1})$ is derived in section 5. In section 6 we give infinitesimally unitary representations of this quantum algebra. Section 7 is devoted to the investigation of self-adjointness of representation operators.

2. The quantum algebra $U_q(su_{1,1})$

We fix a complex number q which does not coincide with a root of unity and give the elements H, E_+, E_- obeying the commutation relations

$$[H, E_+] = E_+ \qquad [H, E_-] = -E_- \tag{1}$$

$$[E_+, E_-] = \frac{q^H - q^{-H}}{q^{1/2} - q^{-1/2}} \tag{2}$$

where $q = \exp h$. The associative algebra A generated by H, E_+, E_- is called a deformation of the universal enveloping algebra $U(sl_2)$ of the Lie algebra $sl(2,\mathbb{C})$.

The structure of a Hopf algebra is introduced into A (Vaksman and Soibelman 1988). The algebra A with this structure is called the quantum algebra $U_q(sl_2)$. It consists of elements which are polynomials of E_+, E_- and finite or infinite series of H. In order to avoid infinite series, instead of H, one considers $k = q^{H/2}$ and $k^{-1} = q^{-H/2}$. This leads to quadratic relations for E_+, E_-, k, k^{-1} (Vaksman and Soibelman 1988)

$$kE_+k^{-1} = q^{1/2}E_+ \qquad kE_-k^{-1} = q^{-1/2}E_-$$

$$[E_+, E_-] = \frac{k^2 - k^{-2}}{q^{1/2} - q^{-1/2}} \qquad kk^{-1} = k^{-1}k = 1.$$

The centre of the algebra $U_q(sl_2)$ is generated by one Casimir element

$$C = E_-E_+ + \left(\frac{q^{(2H+1)/4} - q^{(2H+1)/4}}{q^{1/2} - q^{-1/2}}\right)^2. \tag{3}$$

This element commutes with all elements of $U_q(sl_2)$.

One can introduce *-structures into the Hopf algebra $U_q(sl_2)$ which turn this algebra into *-Hopf algebras. They are q-analogues of real forms of the complex Lie algebra $sl(2, \mathbb{C})$. If $q \in \mathbb{R}$, then the *-structure generated by the relations

$$H^* = H \qquad E_+^* = -E_- \qquad E_-^* = -E_+$$

gives the quantum algebra $U_q(su_{1,1})$ which is an analogue of the real form $su_{1,1}$ of the Lie algebra $sl(2, \mathbb{C})$.

By a linear representation T of the algebra $U_q(sl_2)$ we mean a homomorphism of $U_q(sl_2)$ into the algebra of linear operators (bounded or unbounded) on a Hilbert space, defined on an everywhere dense invariant subspace D, such that the operator $T(H)$ can be diagonalized and has a discrete spectrum. Such representations of $U_q(sl_2)$ lead to linear representations of the associative algebra $U_q(su_{1,1})$ which in general are not representations of the *-Hopf algebra $U_q(su_{1,1})$.

To determine a representation T of $U_q(sl_2)$ it is sufficient to give the operators $T(E_+)$, $T(E_-)$, $T(H)$ for which relations (1) and (2) are fulfilled on an everywhere dense subspace D. If in addition the equalities

$$T(H)^* = T(H) \qquad T(E_+)^* = T(E_-) \tag{4}$$

are satisfied on D, then T is called an infinitesimally unitary representation of the associative algebra $U_q(su_{1,1})$. In this case T is a representation of the quantum algebra (of the *-Hopf algebra) $U_q(su_{1,1})$ which is also called a *-representation. Below, dealing with infinitesimally unitary representations of $U_q(su_{1,1})$, we shall omit the word 'infinitesimally'.

In the papers by Klimyk and Groza (1989), Masuda *et al* (1990) and Vaksman and Korogodsky (1990) representations T (unitary and non-unitary) are considered for which the spectrum of the operator $T(H)$ consists of integers or half-integers. In this paper we deal with representations of $U(su_{1,1})$ for which this condition may not hold. In this way we are led to infinite-dimensional representations of $U_q(su_{1,1})$ which are parametrized by two complex numbers. They are a q-analogue of representations of the universal covering group $\widetilde{SU}(1,1)$ for the Lie group SU(1,1). However, in the quantum case there are some peculiarities which are absent in the classical case.

3. The representations $T_{a\epsilon}$

Let ϵ be a fixed complex number and let V_ϵ be a complex Hilbert space with the orthonormal basis

$$\{|m\rangle;\ m = n + \epsilon,\ n = 0, \pm 1, \pm 2, \ldots\}. \tag{5}$$

For every complex number a we construct the representation $T_{a\epsilon}$ of the associative algebra $U_q(sl_2)$ (and of the associative algebra $U_q(su_{1,1})$) in the Hilbert space V_ϵ defined by the equations

$$H|m\rangle = m|m\rangle \tag{6}$$

$$E_+|m\rangle = [-a+m]|m+1\rangle \qquad E_-|m\rangle = [-a-m]|m-1\rangle \tag{7}$$

where $[b]$ denotes a q-number

$$[a] = (q^{b/2} - q^{-b/2})(q^{1/2} - q^{-1/2})^{-1}.$$

A direct verification shows that the operators (6) and (7) are defined on the everywhere dense subspace D of the Hilbert space V_ϵ consisting of finite linear combinations of the basis elements (5), transform V into V, and satisfy relations (1) and (2) on D. For the Casimir operator $T_{a\epsilon}(C)$ we have

$$T_{a\epsilon}(C)|m\rangle = [a+1/2]^2|m\rangle. \tag{8}$$

There exist equivalence relations in the set of representations $T_{a\epsilon}$. First of all, it is seen from equations (6) and (7) that matrices of the representations $T_{a\epsilon}$ and $T_{a,\epsilon+k}$ coincide for $k \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. For this reason we restrict ourselves to the case

$$0 \leqslant \mathrm{Re}\,\epsilon < 1. \tag{9}$$

The next type of equivalence relations appear because of the periodicity of the function $w(z) = [z]$. We set $q = \exp h,\ h \in \mathbb{R}$. Then the function

$$w(z) = [z] = (q^{z/2} - q^{-z/2})(q^{1/2} - q^{-1/2})^{-1} \tag{10}$$

is periodic with period $4\pi\mathrm{i}/h$. Therefore, it follows from (6) and (7) that

$$T_{a\epsilon} = T_{a+4\pi\mathrm{i}k/h,\epsilon} \qquad \text{for } k \in \mathbb{Z}. \tag{11}$$

For the function (10) we also have

$$w(z) = -w(z + 2\pi\mathrm{i}/h).$$

For this reason, replacement of a by $a+2\pi\mathrm{i}/h$ in equations (6) and (7) leads to the equations

$$H|m\rangle = m|m\rangle \tag{12}$$

$$E_+|m\rangle = -[-a+m]|m+1\rangle \qquad E_-|m\rangle = -[-a-m]|m-1\rangle. \tag{13}$$

They give the representation of $\mathrm{U}_q(\mathrm{su}_{1,1})$ equivalent to the representation $T_{a\epsilon}$ and the equivalence operator is diagonal with respect to the basis $\{|m\rangle\}$ with numbers ± 1 on the main diagonal. Thus

$$T_{a\epsilon} \sim T_{a+2\pi k\mathrm{i},\epsilon} \qquad k \in \mathbb{Z}. \tag{14}$$

If $q = \exp \mathrm{i}h, h \in \mathbb{R}$, then

$$T_{a\epsilon} = T_{a+4\pi k,\epsilon} \qquad \text{for } k \in \mathbb{Z}$$

$$T_{a\epsilon} \sim T_{a+2\pi k,\epsilon} \qquad \text{for } k \in \mathbb{Z}.$$

If $q = \exp\,(h_1 + \mathrm{i}h_2)$ where h_1 and h_2 are non-vanishing real numbers, then the function $w(z) = [z]$ has no periodicities and in this case we do not have analogous equivalences for the representations $T_{a\epsilon}$.

The representations $T_{a\epsilon}$ and $T_{-a-1,\epsilon}$ are also equivalent if they are irreducible. We discuss these equivalences below.

4. Realizations of the representations $T_{a\epsilon}$

If $f(z)$ is a function of a complex variable z, the operator D_z defined by the equation

$$D_z f(z) = \frac{f(q^{1/2}z) - f(q^{-1/2}z)}{z(q^{1/2} - q^{-1/2})}$$

gives the so-called q-differentiation. We consider below functions $f(z, \bar{z})$ depending on z and $\bar{z}$. For these functions we distinguish the q-differentiations D_z and $D_{\bar{z}}$.

Let $W_{a\epsilon}$ be the linear space of complex functions $F(z)$ infinitely q-differentiable at all points except possibly for the point $z = 0$ and such that:

(a) For all positive numbers b the equality $F(bz) = b^{2a}F(z)$ is satisfied.
(b) $F(e^{i\pi}z) = e^{2i\pi\epsilon}F(z)$.

The homogeneity condition $F(bz) = b^{2a}F(z)$ means that a function $F(z)$ is uniquely determined by its values on the sphere S^1 in the complex space $\mathbb{C}$ (Vilenkin and Klimyk 1991, section 6.4.1). Namely, if $z_0 \in S^1$ is a point on the line connecting z with the point $0 \in \mathbb{C}$, then

$$F(z) = \left|\frac{z}{z_0}\right|^{2(a-\epsilon)} \left(\frac{z}{z_0}\right)^{2\epsilon} F(z_0).$$

The space $W_{a\epsilon}$ may be realized as the space of functions

$$f(e^{i\theta}) = e^{-i\epsilon\theta}F(e^{i\theta/2}). \tag{15}$$

These functions are uniquely determined by the functions F. We define the scalar product

$$(f_1, f_2) = \frac{1}{2\pi}\int_0^{2\pi} f_1(e^{i\theta})\overline{f_2(e^{i\theta})}\,d\theta \tag{16}$$

in the space of functions $f(e^{i\theta})$ and close it with respect to the norm $\|f\| = (f, f)^{1/2}$. As a result, we obtain the Hilbert space $L^2(0, 2\pi)$. The scalar product (16) can be transferred into the space $W_{a\epsilon}$ which can also be closed to obtain a Hilbert space.

Let us consider $F(z)$ as a function of z and $\bar{z}$. Then the functions

$$F_m(z, \bar{z}) = z^{a+m}\bar{z}^{a-m} \qquad m = \epsilon + n \qquad n \in \mathbb{Z} \tag{17}$$

form a basis in $W_{a\epsilon}$ orthogonal with respect to the scalar product defined.

Direct evaluations show that the operators

$$T'_{a\epsilon}F(z, \bar{z}) = q^{-a/2}F(q^{1/2}z, \bar{z}) = q^{a/2}F(z, q^{-1/2}\bar{z})$$

$$T'_{a\epsilon}(E_-)F(z, \bar{z}) = -\bar{z}D_z F(z, \bar{z}) = -\bar{z}\frac{F(q^{1/2}z, \bar{z}) - F(q^{-1/2}z, \bar{z})}{z(q^{1/2} - q^{-1/2})}$$

$$T'_{a\epsilon}(E_+)F(z, \bar{z}) = -zD_{\bar{z}} F(z, \bar{z}) = -z\frac{F(z, q^{1/2}\bar{z}) - F(z, q^{-1/2}\bar{z})}{\bar{z}(q^{1/2} - q^{-1/2})}$$

defined on the space $W_{a\epsilon}$ satisfy commutation relations (1) and (2). We also have

$$T'_{a\epsilon}(k)z^{a+m}\bar{z}^{a-m} = q^{m/2}z^{a+m}\bar{z}^{a-m}$$

$$T'_{a\epsilon}(E_-)z^{a+m}\bar{z}^{a-m} = -[a+m]z^{a+m-1}\bar{z}^{a-m+1}$$

$$T'_{a\epsilon}(E_+)z^{a+m}\bar{z}^{a-m} = -[a-m]z^{a+m+1}\bar{z}^{a-m-1}$$

that is, the representation $T'_{a\epsilon}$ is given by equations (6) and (7) with respect to the basis (17). This means that it is equivalent to the representation $T_{a\epsilon}$.

5. Irreducible representations

Irreducibility of the representations $T_{a\epsilon}$ is analysed in the same way as in the case of the Lie group SU(1,1) (Lang 1975). For this reason we shall omit details in our reasoning. In the carrier space of the representation $T_{a\epsilon}$ invariant subspaces appear because of the vanishing of some of the coefficients

$$[-a+m] = [-a+\epsilon+n] \qquad [-a-m] = [-a-\epsilon-n] \qquad n \in \mathbb{Z}$$

from equations (6) and (7). This leads to the following theorem:

Theorem 1. The representation $T_{a\epsilon}$ of the algebra $U_q(sl_2)$ (and of the algebra $U_q(su_{1,1})$) is irreducible if and only if $a \not\equiv \epsilon$ (mod $\mathbb{Z}$) and $a \not\equiv -\epsilon$ (mod $\mathbb{Z}$). If $\epsilon = 0$ or $\epsilon = \frac{1}{2}$ then these inequalities are replaced by one condition $a \not\equiv \epsilon$ (mod $\mathbb{Z}$).

There exist the equivalence relations $T_{a\epsilon} \sim T_{-a-1,\epsilon}$ in the set of irreducible representations $T_{a\epsilon}$. The equivalence operator is evaluated in the same way as in the classical case (see, for example, Vilenkin and Klimyk (1991), section 6.4.4). For $T_{a\epsilon}$ and $T_{-a-1,\epsilon}$ the equivalence operator A is diagonal with respect to the basis $\{|m\rangle\}$ and its diagonal elements are of the form

$$d_m = \frac{[a+\epsilon+1][a+\epsilon+2]\cdots[a+m]}{[-a+\epsilon][-a+\epsilon+1]\cdots[-a+m-1]}. \tag{18}$$

It is possible to show that we have described all equivalence relations for the representations $T_{a\epsilon}$ of $U_q(su_{1,1})$.

Let us consider reducible representations $T_{a\epsilon}$. Let $\epsilon = 0$ or $\epsilon = \frac{1}{2}$. If $a \equiv \epsilon$ (mod $\mathbb{Z}$) then we denote a by l. For $l \geqslant 0$ the invariant subspace

$$V_\epsilon^f = \text{span}\ \{|m\rangle,\ -l \leqslant m \leqslant l\}$$

exists in V_ϵ. The finite-dimensional irreducible representation T_l of the algebra $U_q(su_{1,1})$ acts on this subspace which is well known in the theory of finite-dimensional representations of $U_q(sl_2)$. The quotient space V_ϵ/V_ϵ^f decomposes into the direct sum of two invariant subspaces

$$V_\epsilon^{l+1} = \text{span}\ \{|m\rangle, l < m\} \qquad V_\epsilon^{-l-1} = \text{span}\ \{|m\rangle, m < -l\}.$$

The irreducible representations of $U_q(su_{1,1})$ acting on these subspaces are denoted by T_l^+ and T_l^- respectively. If $l = -\frac{1}{2}$ then the subspace $V_{1/2}^f$ is absent and $V_{1/2}$ decomposes into the orthogonal sum of two invariant subspaces

$$V_{1/2}^{1/2} = \text{span}\ \{|m\rangle, m \geqslant 1/2\} \qquad V_{1/2}^{-1/2} = \text{span}\ \{|m\rangle, m \leqslant -1/2\}.$$

The irreducible representations $T_{-1/2}^+$ and $T_{1/2}^-$ act on these subspaces respectively.

If $\epsilon = 0$ or $\epsilon = \frac{1}{2}$ and $l < -1/2$, then the invariant subspace exists in V_ϵ which decomposes into the orthogonal sum of two invariant subspaces $\bar{V}_\epsilon^{-l}$ and $\bar{V}_\epsilon^l$, where

$$\bar{V}_\epsilon^{-l} = \text{span}\ \{|m\rangle, m \geqslant -l\} \qquad \bar{V}_\epsilon^l = \text{span}\ \{|m\rangle, m \leqslant l\}.$$

The irreducible representations of $U_q(su_{1,1})$ are defined on these subspaces; they are denoted by $\bar{T}^+_{-l-1}$ and $\bar{T}^-_{l+1}$ respectively. The finite-dimensional representation T_{-l-1} acts on the quotient space $V_\epsilon/(\bar{V}^{-l}_\epsilon + \bar{V}^l_\epsilon)$.

Now let $\epsilon \neq 0$ and $\epsilon \neq 1/2$. If $a \equiv \epsilon \pmod{\mathbb{Z}}$ then $a = \epsilon + n$, where $n \in \mathbb{Z}$. For this reason there exists the invariant irreducible subspace in V_ϵ with the highest vector of weight a. We denote it by V^a_ϵ. One has

$$V^a_\epsilon = \operatorname{span}\{|m\rangle,\ m = a - k,\ k = 0, 1, 2, \ldots\}.$$

The representation induced by $T_{a\epsilon}$ on this subspace is denoted by T^-_a. The representation in the quotient space V_ϵ/V^a_ϵ is denoted by $\bar{T}^+_a$. This representation is also irreducible. If $a \equiv -\epsilon \pmod{\mathbb{Z}}$ then the invariant irreducible subspace

$$V^a_\epsilon = \{|m\rangle,\ m = -a + k,\ k = 0, 1, 2, \ldots\}$$

with the lowest vector of weight $-a$ exists in V_ϵ. We denote the representation induced by $T_{a\epsilon}$ in V^a_ϵ by T^+_{-a-1}. The representation on the quotient space V_ϵ/V^a_ϵ is irreducible and is denoted by $\bar{T}^-_a$.

There are equivalence relations in the set of irreducible representations which are irreducible components of reducible representations $T_{a\epsilon}$. Namely, if $\epsilon = 0$ or $\epsilon = \frac{1}{2}$, then

$$T^+_l \sim \bar{T}^+_{l-1} \qquad T^-_{-l} \sim \bar{T}^-_{-l-1}.$$

In other words, in this case we obtain irreducible representations T^+_l, $l = -\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots$, and T^-_l, $l = \frac{1}{2}, 0, -\frac{1}{2}, -1, \ldots$. The spectrum of the operator $T^+_l(H)$ coincides with $l+1, l+2, l+3, \ldots$ and that for the operator $T^-_l(H)$ is $l-1, l-2, l-3, \ldots$.

If $\epsilon \neq 0$ and $\epsilon \neq \frac{1}{2}$, then the representation T^-_{a+1} is equivalent to the representation $\bar{T}^-_{-a}$ and the representation T^+_{a-1} is equivalent to the representation $\bar{T}^+_{-a}$. Thus, in this case we obtain irreducible pairwise non-equivalent representations T^-_a and T^+_a, where $a \not\equiv 0 \pmod{\mathbb{Z}}$ and $a \not\equiv \frac{1}{2} \pmod{\mathbb{Z}}$. The spectrum of the operator $T^-_a(H)$ coincides with the set of points $a - 1 - k$, $k = 0, 1, 2, \ldots$, and that of the operator $T^+_a(H)$ is $a + 1 + k$, $k = 0, 1, 2, \ldots$.

Thus, we can constructed several classes of irreducible representations of the algebra $U_q(su_{1,1})$. If $\epsilon = 0$ or $\epsilon = \frac{1}{2}$ then these classes are:

(a) The representations $T_{a\epsilon}$, where $\operatorname{Re} a \geqslant -\frac{1}{2}$, $a \not\equiv \epsilon \pmod{\mathbb{Z}}$ and also $0 \leqslant \operatorname{Im} a < 2\pi/h$ if $q = \exp h$, $h \in \mathbb{R}$, and $0 \leqslant \operatorname{Re} a < 2\pi/h$ if $q = \exp ih$, $h \in \mathbb{R}$, (q does not coincide with a root of unity).
(b) The representations T^+_l, T^-_{-l}, $l = -\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots$.
(c) Finite-dimensional irreducible representations.

If $\epsilon \not\equiv 0 \pmod{\mathbb{Z}}$ and $\epsilon \not\equiv \frac{1}{2} \pmod{\mathbb{Z}}$ (let us recall that $0 \leqslant \operatorname{Re} \epsilon < 1$), then we have the following classes of irreducible representations:

(a) The representations $T_{a\epsilon}, a \not\equiv \epsilon \pmod{\mathbb{Z}}$, $a \not\equiv -\epsilon \pmod{\mathbb{Z}}, \operatorname{Re} a \geqslant -\frac{1}{2}$ and also $0 \leqslant \operatorname{Im} a < 2\pi/h$ if $q = \exp h$, $h \in \mathbb{R}$, and $0 \leqslant \operatorname{Re} a < 2\pi/h$ if $q = \exp ih$, $h \in \mathbb{R}$ (h is not a root of unity).
(b) The representations T^+_a, T^-_a.
(c) Finite-dimensional irreducible representations.

The associative algebra $U_q(su_{1,1})$ has no other algebraically irreducible representations. Let T denote an irreducible representation of this algebra. Then the operator $T(C)$ is multiple of the unit operator on the carrier space of the representation T. The operator $T(H)$ can be diagonalized and has a discrete spectrum. Let $|m\rangle$ be an eigenvector of $T(H)$ corresponding to an eigenvalue $m = \epsilon + n$, $n \in \mathbb{Z}$. We construct the vectors

$$\cdots \quad |m-2\rangle = T(E_-)^2|m\rangle \qquad |m-1\rangle = T(E_-)|m\rangle \qquad |m\rangle$$
$$|m+1\rangle = T(E_+)|m\rangle \qquad |m+2\rangle = T(E_+)^2|m\rangle \qquad \cdots.$$

It is proved in the standard manner that the vectors $|m+k\rangle$, $k \in \mathbb{Z}$, are eigenvectors for the operator $T(H)$ corresponding to the eigenvalues $m+k$ respectively. Since

$$T(E_-)T(E_+)|m\rangle = T(C)|m\rangle - d|m\rangle$$

where d is a constant and $T(C)$ is the Casimir operator, then up to constants we have

$$\begin{aligned} T(E_-)|m+i\rangle &= T(E_-)T(E_+)|m+i-1\rangle \\ &= T(C)|m+i-1\rangle - b|m+i-1\rangle = |m+i-1\rangle \\ T(E_+)|m-i\rangle &= T(E_+)T(E_-)|m-i+1\rangle \\ &= T(E_-)T(E_+)|m-i+1\rangle - c|m-i+1\rangle = |m-i+1\rangle. \end{aligned}$$

This means that the spectrum of the operator $T(H)$ is simple. More detailed calculations show that the representation T is equivalent to the representation $T_{a\epsilon}$ if $T(C) = [a+\frac{1}{2}]^2$ and the spectrum of $T(H)$ does not terminate from below or from above. It is equivalent to the representation T_a^+ (to the representation T_a^-) if $T(C) = [a+\frac{1}{2}]^2$ and this spectrum terminates from below (respectively from above) but does not terminate from the other side. If the spectrum of $T(H)$ terminates from both sides, then T is equivalent to a finite-dimensional representation.

6. Unitary representations

Let D be the linear subspace in the carrier Hilbert space V_ϵ of the representation $T_{a\epsilon}$ spanned by the basis vectors $|m\rangle$, $m = \epsilon + n$, $n \in \mathbb{Z}$. We set $q = \exp h$, $h \in \mathbb{R}$. Let us find for which representations $T_{a\epsilon}$ relations (4) are satisfied on D. It is clear that the condition $T(H)^* = T(H)$ means that the spectrum of the operator $T(H)$ is real, that is $0 \leqslant \epsilon < 1$. The condition $T(E_+)^* = -T(E_-)$ means that for all $m = \epsilon + n$, $n \in \mathbb{Z}$, the condition

$$-a + m - 1 = \overline{a+m}$$

must be satisfied, where the bar means complex conjugation. This condition is fulfilled if and only if $a = \mathrm{i}\rho - \frac{1}{2}$, $\rho \in \mathbb{R}$. Thus, the representations $T_{\mathrm{i}\rho-1/2,\epsilon}$, $\rho \in \mathbb{R}$, $0 \leqslant \epsilon < 1$, are unitary. They form the principal unitary series of representations of the quantum algebra $U_q(\mathrm{su}_{1,1})$.

In order to find other unitary representations of $U_q(\mathrm{su}_{1,1})$ we introduce, as in the classical case (see, for example, Vilenkin and Klimyk 1991, section 6.4.6), a new scalar product in carrier spaces of irreducible representations. Namely, we pass from the basis $\{|m\rangle\}$ to the basis $\{|m\rangle'\}$, where

$$|m\rangle = d_m^{1/2}|m\rangle' \equiv \left(\frac{[a+\epsilon+1][a+\epsilon+2]\cdots[a+m]}{[-a+\epsilon][-a+\epsilon+1]\cdots[-a+m-1]}\right)^{1/2}|m\rangle'$$

(see equation (18)), and consider that the basis $\{|m\rangle'\}$ is orthonormal with respect to the new norm. Operators of the representation $T_{a\epsilon}$ are given in this basis by the equations

$$T_{a\epsilon}(H)|m\rangle' = m|m\rangle' \tag{19}$$

$$T_{a\epsilon}(E_+)|m\rangle' = ([a+m+1][-a+m])^{1/2}|m+1\rangle' \tag{20}$$

$$T_{a\epsilon}(E_-)|m\rangle' = -([-a+m-1][a+m])^{1/2}|m-1\rangle'. \tag{21}$$

Operators of the representations T_a^+ and T_a^- are given in the basis $\{|m\rangle'\}$ by the same equations. In the last case values of the parameter m are bounded from below or from above.

As in the classical case, we now verify for which irreducible representations of $U(su_{1,1})$ the relations (4) are satisfied. Such analysis gives us the following additional classes of unitary irreducible representations of $U(su_{1,1})$:

(a) The representations $T_{a\epsilon}$ with $0 \leqslant \epsilon < 1$, where $\epsilon - 1 > a > -\epsilon$ for $\epsilon > \frac{1}{2}$ and $-\epsilon > a > \epsilon - 1$ for $\epsilon < \frac{1}{2}$ (the supplementary series).
(b) The representations $T_{a\epsilon}$, Im $a = \pi/h$, Re $a > -\frac{1}{2}$ (the strange series).
(c) All representations T_a^+, $a \geqslant -\frac{1}{2}$, and T_a^-, $a \leqslant \frac{1}{2}$ (the discrete series).
(d) The zero representation.

Let us remark that in the case of representations of the principal unitary series the transition from the basis $\{|m\rangle\}$ to the basis $\{|m\rangle'\}$ is given by the diagonal unitary matrix, that is $|d_m| = 1$. For this reason operators of representations of this series satisfy relations (4) if they are presented in the matrix form with respect to the basis $\{|m\rangle'\}$, that is, by equations (19)–(21). We also note that the representations T_a^+ and T_a^- and their realizations were constructed by Kalnins *et al* (1992).

It is possible to show that the unitary irreducible representations listed above, exhaust all irreducible unitary representations of $U_q(su_{1,1})$.

Let $q = \exp ih$, $h \in \mathbb{R}$, is not a root of unity. In the same way as in the previous case, it is shown that now we have the following series of irreducible unitary representations of $U_q(su_{1,1})$:

(a) The representations $T_{a\epsilon}$, $0 \leqslant \epsilon < 1$, $a = i\rho - \frac{1}{2}$, $\rho \in \mathbb{R}$ (the principal unitary series).
(b) The representations $T_{a\epsilon}$, $0 \leqslant \epsilon < 1$, Re $a = \pi/h$ (the strange series).
(c) The zero representation.

The discrete series representations are absent in this case.

If $q = \exp(h_1 + ih_2)$, $h_1, h_2 \in \mathbb{R}$, $h_1 \neq 0$, $h_2 \neq 0$, then the algebra $U_q(su_{1,1})$ has no irreducible unitary representations except for the zero one.

7. Self-adjoint extension of operators

A natural problem appears in the theory of representations of Lie algebras concerning integrability of these representations to representations of the corresponding Lie groups. This problem is solved for finite-dimensional Lie algebras (see, for example, Barut and Raczka 1977). This problem is reduced to proving self-adjointness or the existence of self-adjoint extensions for operators corresponding to elements of Lie algebras.

We need to know about existence of self-adjoint extensions of representation operators for Lie algebras when we try to embed physical observables (Hamiltonians, operators of momentum, and other operators) into representations of these algebras. The analogous problem appears for representations of quantum algebras. Therefore, the problem of self-adjointness or of existence of self-adjoint extensions for operators of representations of quantum algebras is also important.

In the case of unitary representations T of the quantum algebra (the *-Hopf algebra) $U_q(su_{1,1})$ the operators

$$T(M) = T(E_+ + E_-) \qquad T(N) = T(iE_+ - iE_-) \tag{22}$$

are symmetric. We shall show that for $q = \exp h$, $h \in \mathbb{R}$, these operators admit self-adjoint extensions for every irreducible unitary representation.

In the case of representations of Lie algebras, self-adjointness of operators is easily proved with the help of results on analytical vectors (Nelson 1959). A vector $|v\rangle$ is analytical for an operator A if the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} t^n \| A^n |v\rangle \|$$

converges for some $t > 0$. If the set V of analytical vectors of a symmetric operator A is everywhere dense in a Hilbert space where A operates, then this operator has self-adjoint extensions. In the case of representations of the Lie algebra su(1,1) (when $q = 1$) the operators (22) have a dense set of analytical vectors containing the basis vectors $|m\rangle$, and therefore they have self-adjoint extensions. A good example of the investigation of existence of self-adjoint extensions of representation operators is given in the paper by Mickelsson and Niederle (1973).

The corresponding results for the operators $T(M)$ and $T(N)$ of representations of the quantum algebra $U_q(su_{1,1})$, $q \neq 1$, are not valid. The q-analogue of the Stirling equation for $[n]! \equiv [1][2]\cdots[n]$ is of the form

$$[n]! \sim q^{-n(n-1)/4}(1-q)^{-n} \exp(-C_q)$$

where $0 < q < 1$ and

$$C_q = -\sum_{k=0}^{\infty} \ln(1 - q^{k+1})$$

(Nikiforov and Uvarov 1988). The evaluations similar to those of Mickelsson and Niederle (1973) show that for large n one has

$$\| T(M)^n |m\rangle \| \sim c q^{-n(n-1)/4} (1-q)^{-n}$$

if $0 < q < 1$, where c is a constant. Therefore, the basis vectors $|m\rangle$ are not analytical for the operator $T(M)$. If $q > 1$ then replacing q by q^{-1}, it is shown that in this case the vectors $|m\rangle$ are also not analytical for $T(M)$.

To investigate the self-adjointness of $T(M)$ we apply the method of Jacobi matrices (Berezanskij 1968). Jacobi matrices are determined by difference operators L of the second order on the real half-axis. These operators are represented in the corresponding basis by matrices of the type

$$\begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{23}$$

Let us assume that $a_j > 0$ for all j. Then the operator L has deficiency indices (0,0) or (1,1) (Berezanskij 1968, chapter 7). In the first case the closure of the operator L, defined on the dense set of finite vectors (finite linear combinations of basis elements for which L has the Jacobi form (23)), is self-adjoint. In the second case L has self-adjoint extensions.

The elements $|m\rangle$ of the carrier spaces of the representations T_a^+ are labelled by the numbers $m = a+1, a+2, \ldots$. The operators of these representations are of the form

$$T_a^+(H)|m\rangle = m|m\rangle$$

$$T_a^+(M)|m\rangle = b(m-1)|m-1\rangle + b(m)|m+1\rangle$$

$$T_a^+(N)|m\rangle = ib(m-1)|m-1\rangle - ib(m)|m+1\rangle$$

where $b(m) = ([m-a][a+m+1])^{1/2}$. It is easy to see that

$$b^2(m) = [m-a][a+m+1] = \frac{\cosh \frac{1}{2}h(2m-1) - \cosh \frac{1}{2}h(2l+1)}{2\sinh^2 \frac{1}{2}h}.$$

One can see from here that $b^2(m) > 0$ for all $m = a+k$, $k = 1, 2, 3, \ldots$. Thus, the operator $T_a^+(M)$ is self-adjoint or it has a self-adjoint extension. Existence of a self-adjoint extension for the operator $T_a^-(M)$ is proved in the same way.

In the case of unitary representations of other series the operator $T(M)$ is not of the form (23) since the spectrum of the operator $T(H)$ for these representations does not terminate from below or above. In this case the method of doubling (Berezanskij 1968) is used. We introduce the notation

$$|\mathbf{m}\rangle = \begin{vmatrix} |-m-1\rangle \\ |m\rangle \end{vmatrix} \qquad B(m) = \begin{pmatrix} a(-m-2) & 0 \\ 0 & a(m) \end{pmatrix}.$$

The operator $T(M)$ for the unitary representations under consideration can be represented in the form

$$T(M)|\mathbf{m}\rangle = B(m-1)|\mathbf{m}-\mathbf{1}\rangle + B(m)|\mathbf{m}+\mathbf{1}\rangle$$

where $m = 0, 1, 2, \ldots$. Hence it has the form of a Jacobi matrix with operator coefficients

$$T(M) = \begin{pmatrix} 0 & B(1) & 0 & 0 & \cdots \\ B(1) & 0 & B(2) & 0 & \cdots \\ 0 & B(2) & 0 & B(3) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where 0 denotes the zero 2×2 matrix. If elements of the matrices $B(m)$ are real, then the operator $T(M)$ has deficiency indices (0,0), (1,1), or (2,2) and it has a self-adjoint extension (Berezanskij 1968, chapter 7).

The basis elements of carrier spaces of representations of the supplementary series are labelled by the numbers $m = \epsilon + n$, $n \in \mathbb{Z}$, and

$$T(M)|m\rangle' = b(m)|m+1\rangle' + b(m-1)|m-1\rangle'$$

where

$$b^2(m) = [m-a][a+m+1] = \frac{\cosh \frac{1}{2}h(2m+1) - \cosh \frac{1}{2}h(2a+1)}{2\sinh^2 \frac{1}{2}h}.$$

We have $\epsilon - 1 > a > -\epsilon$ for $\epsilon > \frac{1}{2}$ and $-\epsilon > a > \epsilon - 1$ for $\epsilon < \frac{1}{2}$. Therefore, for all admissible values of the parameters ϵ and m we have $b^2(m) > 0$. This means that the operator $T(M)$ for representations of the supplementary series admits a self-adjoint extension.

For the representations $T_{a\epsilon} \equiv T_{i\rho-1/2,\epsilon}$ of the principal unitary series we have

$$b^2(m) = [m-a][a+m+1] = \frac{\cosh \frac{1}{2}h(2m+1) - \cos \frac{1}{2}h\rho}{2\sinh^2 \frac{1}{2}h} \geqslant 0.$$

Therefore, the operator $T(M)$ for these representations admits a self-adjoint extension.

For the representations $T_{r+i\pi/h}$, $r \in \mathbb{R}$, of the strange series we obtain

$$b^2(m) = \frac{\cosh \frac{1}{2}h(2m+1) + \cosh \frac{1}{2}h(2r+1)}{2\sinh^2 \frac{1}{2}h} \geqslant 0.$$

Thus, in this case the operator also has a self-adjoint extension.

To prove that the operator $T(N)$ of unitary representations also admits self-adjoint extensions, we pass in equations (6) and (7) from the basis $\{|m\rangle\}$ to the basis $\{|m\rangle''\}$, where $|m\rangle'' = i^m|m\rangle$. Then we have

$$T(H)|m\rangle'' = m|m\rangle'' \qquad T(E_+)|m\rangle'' = -i[m-a]|m+1\rangle'' \tag{24}$$

$$T(E_-)|m\rangle'' = i[-a-m]|m-1\rangle''. \tag{25}$$

Representations given by these equations are equivalent to the corresponding representations given by equations (6) and (7). The operator $T(N)$ in the basis $\{|m\rangle''\}$ is given by the equation for the operator $T(M)$ in the basis $\{|m\rangle\}$. We proved that the operator $T(M)$ has a self-adjoint extension. Therefore, the operator $T(N)$ also admits a self-adjoint extension.

It is shown by Berezanskij (1968) that self-adjoint extensions of symmetric operators representable by Jacobi matrices with usual or operator coefficients are related to orthogonal polynomials. Values of deficiency indices and dense subspaces on which a symmetric operator is self-adjoint are determined by these polynomials. They also define the spectrum of the self-adjoint extension. Evaluation of these polynomials for operators $T(M)$ and $T(N)$ will be given in a forthcoming paper.

8. Conclusion

We have found all irreducible representations of the algebra $U_q(su_{1,1})$. They are given by two complex parameters. We separated all infinitesimally unitary representations in the set of irreducible representations. There is the principal unitary series, the supplementary series, the strange series and the discrete series of infinitesimally unitary representations. The strange series disappears when $q \to 1$.

Irreducible representations of $\bar{U}_q(su_{1,1})$ were constructed with the help of the so called standard representations $T_{a\epsilon}$ of this algebra. In the set of representations $T_{a\epsilon}$ there exist equivalence relations which are consequences of the periodicity of the function $w(z) = [z]$. These equivalences lead to the appearance of the Jacobi theta functions in the Plancherel measure when we decompose regular or quasiregular representation of $U_q(su_{1,1})$.

We also considered the problem of self-adjointness of symmetric operators of infinitesimally unitary representations of $U_q(su_{1,1})$. It is shown that the symmetric operators $T(E_+ + E_-)$ and $T(iE_+ - iE_-)$ of irreducible unitary representations admit self-adjoint extensions. Investigations in this area must be continued. In particular, it is necessary to study the self-adjoint extensions of all representation operators corresponding to symmetric elements of $U_q(su_{1,1})$.

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